Numerical study of stability and transient phenomena of Poiseuille flows in ducts of square cross-sections

A.V. Boiko, Yu.M. Nechepurenko
Khristianovich Institute of Theoretical and Applied Mechanics SB RAS
630090, Novosibirsk, Russia
Institute of Numerical Mathematics RAS
119333, Moscow, Russia

This work is devoted to the temporal stability problems for Poisseuille flows in ducts of square cross-sections. Results of previous theoretical and numerical studies [1, 2] indicate that the flow in such a duct is stable to any infinitesimal disturbances, i.e. velocity and pressure deviations from the steady state, that ex facto contradicts to existing experimental data. Comparison of this situation with circular pipe flow [3] leads to an assumption that a non-orthogonality of eigenmodes is one of the main factors of the disturbance kinetic energy density growth in finite time intervals that triggers laminar-turbulent transition in real flows. Most growing, in this sense, disturbances are usually called optimal. Their knowledge is important in courses of laminar-turbulent transition and robust flow control investigations.

We considered the dimensionless linearized Navier-Stokes equations for disturbances. Assuming a harmonic dependence of the disturbances on streamwise space variable, we transformed the equations to a system of partial differential equations for velocity components and pressure which depended only on spanwise space variables and time. A new classification of solutions of the system based on their symmetries was proposed. Stability characteristics as location of leading eigenvalues, shape of eigenmodes, values of maximal energy growth and critical Reynolds numbers were computed.

After approximating the system with respect to space variables and splitting the result into systems of ordinary differential and algebraic equations (ODAE) corresponding to different symmetries we used a new order reduction technique for the systems. The reduction results to systems of ODE of halved dimensions of the original systems, so that the final dimensions coincide with the dimensions of the solution spaces. The reduction was proposed in [4] for linearized and discretized Navier—Stokes equations for disturbances in ducts of arbitrary constant cross-sections. It is based on orthogonal matrix transformations and does not deteriorate the numerical stability of the original system of ODAE. Then, a modal subspace expansion of the solution of each reduced system was used which substantially improved numerical stability and reduced computational costs compared with modal expansion.

Problem formulation. Consider in dimensionless Cartesian coordinates $x, y, z$ the Poisseuille flow of viscous incompressible fluid in duct $\{(x,y,z): -l < x < l, -1 < y < 1, -\infty < z < \infty\}$ of square cross-section $\Sigma=\{(x,y): -l < x < l, -1 < y < 1\}$. The corresponding dimensionless Poisseuille flow velocity distribution can be found by solving the equation

$$\frac{\partial^2 \tilde{W}}{\partial x^2} + \frac{\partial^2 \tilde{W}}{\partial y^2} = -1$$

in $\Sigma$ with homogeneous boundary conditions and, then, by applying $\tilde{W}(x,y) = \tilde{W}(x,y)/\tilde{W}(0,0)$. Denoting dimensionless disturbance velocity components along $x, y, z$ and pressure as $u', v', w', p'$, respectively, and the Reynolds number as $Re$, the linearized viscous incompressible flow equations for the disturbance with homogeneous boundary conditions for $u', v', w'$ at the duct walls can be written as:
The problem of temporal linear stability of the flow under consideration consists of investigation of the following waveform solutions

\[
\begin{align*}
&u'(x, y, z, t) = v \exp(i\alpha z), \\
v'(x, y, z, t) = v(x, y, t) \exp(i\alpha z), \\
p'(x, y, z, t) = p(x, y, t) \exp(i\alpha z),
\end{align*}
\]  

(2)

where \(\alpha\) is a given real constant (wave number). If \(\alpha = 0\) then solution (2) is real. Otherwise it is complex but its real part satisfies the equations (1) as well. This real part is usually represented as

\[
\begin{align*}
&\Delta u' = \frac{\partial u'}{\partial t}, \\
&\Delta v' = \frac{\partial v'}{\partial t}, \\
&\Delta p' = \frac{\partial p'}{\partial t}.
\end{align*}
\]  

(1)

The velocity components and pressure of the real part of the disturbance of the form (2) experience harmonic oscillations along \(z\) at fixed \(t\) with in general different amplitudes and phases at different points of the duct cross-section. The mean kinetic energy of the disturbance per unit duct length can be represented as

\[
\frac{\alpha}{4\pi} \int_0^{2\pi} \left[ |\text{Real } u'|^2 + |\text{Real } v'|^2 + |\text{Real } w'|^2 \right] dxdydz = \frac{1}{4\pi} \int_0^{2\pi} |\text{Real } u|^2 + |\text{Real } v|^2 + |\text{Real } w|^2 dxdydz. 
\]  

(3)

Substituting (2) into (1) we obtain the following system for \(v\) and \(p\):  

\[
\frac{\partial v}{\partial t} = Jv + Gp, Fv = 0,
\]

where

\[
J = \begin{pmatrix}
L & 0 & 0 \\
0 & L & 0 \\
-W_x & -W_y & L
\end{pmatrix},
\]

\[
G = \begin{pmatrix}
-\partial/\partial x \\
-\partial/\partial y \\
-i\alpha
\end{pmatrix},
\]

\[
L = -i\alpha W + \frac{1}{\text{Re}} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \alpha^2 \right),
\]

\[
F = \left( \partial/\partial x, \partial/\partial y, i\alpha \right), W_x = \partial W/\partial x, W_y = \partial W/\partial y.
\]

Among solutions of the system (4) the solutions of the form

\[
v(t) = v_* \exp(\lambda_* t), p(t) = p_* \exp(\lambda_* t),
\]

(5)

where \(\lambda_*\) is a finite eigenvalue of the operator pencil

\[
P(\lambda) = \begin{pmatrix}
\lambda - J & -G \\
F & 0
\end{pmatrix},
\]

and \((v_*,p_*)^T\) is the corresponding eigenfunction, are of particular interest. Such solutions are called modes of (4). If pencil \(P(\lambda)\) has a finite eigenvalue of multiplicity one with the largest real part, then mode (5) is called the leading mode.

Stability of the flows in rectangular ducts was studied numerically in [1, 2]. In [1] equations (4) were used directly, that after approximation with respect to space, leads to a system of ODAE of double algebraic dimension, than the real dimension of the processes under study. In [2] a preliminary analytic transformation of (4) to a system of two equations for wall-normal velocity components valid at \(\alpha \neq 0\) was used. The transformation enlarges the differential order of the equations to four that leads to computations with ill-conditioned matrices.
**Algebraic reduction.** Let system (4) have already been approximated in $x$ and $y$ yielding the following system of ODAE:

$$\frac{d^2 v}{dt^2} = J v + G p, \quad F v = 0$$

(6)

where $v \in \mathbb{C}^n$ and $p \in \mathbb{C}^{n_p}$ are vectors of discrete velocity components and pressure, respectively ($n_p < n_v$); $C$ and $J$ are square matrices of order $n_v$; $G$ and $F$ are rectangular matrices of sizes $n_v \times n_p$ and $n_p \times n_v$, respectively. Here $C$ is real while $J$, $G$ and $F$ are real at $\alpha = 0$ and complex otherwise.

Let (6) inherit from (4) the possibility to eliminate pressure from momentum equations with the help of the continuity equation. It implies that matrices $C$ and $FC^{-1}G$ are nonsingular or (6) can be reduced to a system of the same form and with the matrices possessing this property by eliminating some columns of $G$ and rows of $F$. Consider the following initial value problem for system (6) with nonsingular matrices $C$ and $FC^{-1}G$: given $v^0 \in \mathbb{C}^n$ such that $Fv^0 = 0$, find $v(t)$ and $p(t)$, satisfying (6) at $t=0$ and $v(0) = v^0$.

Let

$$G = U \begin{pmatrix} R \\ 0 \end{pmatrix}, \quad F^* = V \begin{pmatrix} T \\ 0 \end{pmatrix}$$

be the QR-decompositions, where $U$ and $V$ are unitary matrices of order $n_v$, while $R$ and $T$ are upper triangular of order $n_p$. Consider matrices $\tilde{J} = U^* J V$, $\tilde{C} = U^* C V$ partitioned as follows:

$$\tilde{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \quad U = (U_1 \quad U_2), V = (V_1 \ V_2),$$

where block $(k,l)$ is a matrix of size $n_k \times n_l$ ($k, l = 1, 2$), blocks $U_1$ and $V_1$ are matrices of size $n_v \times n_l$, and $n_1 = n_p$, $n_2 = n_v - n_p$.

As it was shown in [4], the solution of the initial value problem exists, unique and can be written as

$$v(t) = V_2 \exp(tH)V_1^* v_0, \quad p(t) = R^{-1} \begin{pmatrix} \tilde{C}_{12} H - \tilde{J}_{12} \end{pmatrix} \exp(tH) V_2^* v_0,$$

(7)

where $H = \tilde{C}_{22}^{-1} \tilde{J}_{22}$, and that matrix $H$ has the same spectrum as the finite spectrum of the matrix pencil corresponding to system (6). Representation (7) is highly resistant to rounding errors in floating point computations [4].

**Spectral decomposition.** Let $\lambda(H) = \lambda(H)_1 \cup \lambda(H)_2$ be a given spectrum dichotomy into two nonempty nonintersecting subsets and

$$H = (Q_1 \quad Q_2) S(Q_1 \quad Q_2)^*, \quad S = \begin{pmatrix} S_1 & S_{12} \\ 0 & S_2 \end{pmatrix}$$

be the corresponding Schur decomposition, i.e. $[Q_1, Q_2]$ is a unitary matrix of order $n_v - n_p$, $S_1$ and $S_2$ are upper triangular matrices of orders $l$ and $n_v - n_p - l$, respectively, and $\lambda(S_j) = \lambda(H)_j$, $j=1, 2$.

Then,

$$v(t) = V_2 Q_1 \exp(tS_1) \chi, \quad p(t) = R^{-1} \begin{pmatrix} \tilde{C}_{12} H - \tilde{J}_{12} \end{pmatrix} Q_1 \exp(tS_1) \chi$$

(8)

with an arbitrary $l$-component vector $\chi$ is the general solution of system (6) belonging to the subspace of modes corresponding to $\lambda(H)_1$.

**Optimal disturbances.** Denote the solution of the initial value problem under consideration as $v(t; v^0)$. Let (6) be Lyapunov asymptotically stable, i.e. $v(t; v^0) \to 0$ at $t \to \infty$ for any allowable initial value $v^0$. This is equivalent to location of all eigenvalues of $H$ strictly in the left half-plane.

Let $\mathcal{E}(v) = (Ev, v)$ be a discrete analog of the mean kinetic energy density (3), where $E$ is a Hermitian positive-definite matrix accounting for an uneveness of discretization in $x$ and $y$. The initial value $v^0$ is called the optimal disturbance, if maximum of $\mathcal{E}(v(t; v^0))$ in $0 \leq t < \infty$ is the largest.
among all allowable initial values with $\mathcal{E}(v^0) = 1$. We further restrict the set of optimal disturbances to those which reach maximum at smallest value of $t$, i.e., $t_{\text{opt}} = \min \arg \max \{\gamma(t) : t \geq 1\}$ where

$$
\gamma(t) = \max \{\mathcal{E}(v(t,v^0)) : \mathcal{E}(v^0) = 1\}
$$

is the maximal possible amplification of the kinetic energy density at time $t$.

Using first formula in (7) it can be easily shown that $\mathcal{E}(v(t,v^0))$ can be written as

$$
\mathcal{E}(v(t,v^0)) = \| \exp(t\widetilde{H}) \mathcal{V}^0 \|_2^2
$$

where $\widetilde{H} = L^*H^*L$, $\mathcal{V}^0 = L^*v^0$ and $L$ is the lower triangular factor in the Cholesky decomposition $V^*_2E_2 = LL^*$. Hence, $\gamma(t) = \| \exp(t\widetilde{H}) \mathcal{V}^0 \|_2^2$ and computation of $t_{\text{opt}}$ is reduced to computations of the norm of the matrix exponential $\exp(t\widetilde{H})$ at different $t$. Any optimal disturbance can be expressed as $v^0_{\text{opt}} = V^*_2L^*\mathcal{V}^0_{\text{opt}}$, where $\mathcal{V}^0_{\text{opt}}$ is normalized singular vector corresponding to the largest singular value of matrix $\exp(t_{\text{opt}}\widetilde{H})$. Converse is also valid. Thus, to find all optimal disturbances it is sufficient to compute $t_{\text{opt}}$ and find the orthonormal basis in the subspace of singular vectors of matrix $\exp(t_{\text{opt}}\widetilde{H})$ corresponding to its largest singular value.

Using (8) instead of (7) it is possible to solve the problem of optimal disturbances from the subspace of most slowly decaying modes. The results of numerical experiments below show that such an optimal disturbance reaches the optimal disturbance from the whole space at relatively small $l$ and experiences virtually no changes at larger $l$. This fact can be used to significantly reduce the cost of the optimal disturbance computations.

**Condition for energy growth.** The disturbance growth rate can be estimated as

$$
\frac{d}{dt} \mathcal{E}(v(t,v^0)) = \frac{d}{dt} \| \exp(t\widetilde{H}) \mathcal{V}^0 \|_2^2 \leq h_{\text{max}} \| \exp(t\widetilde{H}) \mathcal{V}^0 \|_2^2 = 2h_{\text{max}} \mathcal{E}(v(t,v^0))
$$

where $h_{\text{max}}$ denotes the maximal eigenvalue of the Hermitian matrix $(\widetilde{H} + \widetilde{H}^*)/2$. Moreover, the upper bound is attained at $t=0$ when $\mathcal{V}^0$ is an eigenvector of the matrix $(\widetilde{H} + \widetilde{H}^*)/2$ which corresponds to its maximal eigenvalue $h_{\text{max}}$. Hence, the condition for energy growth is $h_{\text{max}} > 0$ and the critical Reynolds number $Re_{\text{crit}}$ below (no energy growth is possible at all if $Re \leq Re_{\text{crit}}$) can be found as the Reynolds number at which $h_{\text{max}} = 0$.

**Symmetries.** Denote

$$f_{\text{ex}}(x,y) = (f(x,y)\delta f(-x,y)+f(x,y)\delta f(-x,y))/4, f^*(x,y) = f(y,x).$$

Taking into account that the Poiseuille flow velocity distribution is an even function in spanwise coordinates it has already been found [2] that if $(u,v,w,p)$ is a solution of system (4) then

$(u_{-},v_{+},w_{-},p_{+}), (u_{-},v_{+},w_{+},p_{-}), (u_{+},v_{-},w_{-},p_{-}), (u_{+},v_{+},w_{+},p_{+})$

satisfy the system as well. It takes us possible to focus at solutions $(u,v,w,p)$ possessing the following four symmetries:

**I) $u = u_{-}, v = v_{+}, w = w_{-}, p = p_{+}$**

**II) $u = u_{-}, v = v_{+}, w = w_{+}, p = p_{+}$**

**III) $u = u_{+}, v = v_{-}, w = w_{-}, p = p_{-}$**

**IV) $u = u_{+}, v = v_{-}, w = w_{+}, p = p_{-}$**

Let $(u,v,w,p)$ be a solution of class II or III. Taking into account that the Poiseuille flow velocity distribution is symmetric with respect to line $x=y$ one can prove that

$(u \pm v^*, v \pm u^*, w \pm w^*, p \pm p^*)$

are solutions of the same class as well. So, we can subdivide class II into

**II$\pm$u**

and class III into

**III$\pm$u**
Note that each solution \((u,v,w,p)\) of the system (4) can be represented as a sum of solutions possessing the above six different symmetries \(I, \ II^+, \ II^-, \ III^+, \ III^-\) and \(IV\). Velocities of the solutions of different symmetries are mutually orthogonal. It takes us possible to study the subsets of solutions separately. Moreover, classes \(I\) and \(IV\) are identical up to axes rotation around their origin. Thus, there remain only five different classes of the solutions to study.

**Numerical approximation.** To approximate (4) in \(x\) and \(y\) we applied the collocation technique. The roots \(\xi_i = \cos(\pi i/(2q + 1)), \ i = 1,\ldots,2q\), of the second-kind Chebyshev polynomial \(U_{2m}(\xi)\) for pressure and the same points together with \(\pm 1\) were used for velocities as interpolation knots. Matrices of discrete analogs of derivatives were calculated in MATLAB with functions similar to those described in [6]. Then the symmetry conditions were utilized to split the original system (6) into five subsystems of the same form and smaller dimensions.

**Results.** The numerical technique described above was used to study the development of disturbances. We approximated system (4) with \(q = 21\) so that \(n_v = 1323\) and \(n_p = 441\) for the original system (6). Most of the data are presented for class \(I\) at \(Re = 3000\) and \(\alpha = 0.1\).

<table>
<thead>
<tr>
<th>No.</th>
<th>(I)</th>
<th>(II_+)</th>
<th>(II_-)</th>
<th>(III_+)</th>
<th>(III_-)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.0123 - i0.0739</td>
<td>-0.0147 - i0.0839</td>
<td>-0.0148 - i0.0447</td>
<td>-0.0182 - i0.0636</td>
<td>-0.0150 - i0.0850</td>
</tr>
<tr>
<td>2</td>
<td>-0.0140 - i0.0439</td>
<td>-0.0220 - i0.0732</td>
<td>-0.0174 - i0.0677</td>
<td>-0.0196 - i0.0613</td>
<td>-0.0208 - i0.0403</td>
</tr>
</tbody>
</table>

At first, matrices \(V_2\) and \(H\) were formed and eigenvalues of \(H\) were computed. Fig. 1 shows all \(n_v - n_p = 882\) eigenvalues of \(H\) in the main plot and the leading eigenvalues (with the largest real parts) in the insertion. All eigenvalues are located strictly in the left half-plane and stretched along the real axis. The leading part of the spectrum has certain structure: the eigenvalues form the “branches” characteristic also for semi-infinite duct flow [5]. As seen in Table 1 the first leading eigenvalue of class \(I\) is simple and well separated from the remaining part of the spectrum both in its class and in the whole spectrum; the eigenvalues from the other classes have smaller real parts.
Section II

Fig. 2 shows amplitudes $|u|$, $|v|$, $|w|$ and phases $\varphi_u$, $\varphi_v$, $\varphi_w$ of the first-mode velocity components $u$, $v$, $w$, respectively. The phases were adjusted to symmetrize $\varphi_u$ and $\varphi_v$. The phase jumps of $u$ along $(0,y)$ and $(x,0)$ and of $w$ along $(x,0)$ by $\pi$ are caused by the antisymmetry of $u$ in $x$ and $y$ and $w$ in $y$. Hence, the solution (2) of linearized Navier—Stokes equation (1) corresponding to the leading mode experiences harmonic oscillations along $z$ at fixed $t$ with phase shifts at different cross-section points.

Fig. 2. Amplitudes and phases of the mode velocity components corresponding to the first leading eigenvalue.

Fig. 3 illustrates the behavior of the second leading mode in the same manner. The main qualitative difference with the first leading mode consists in larger number of local amplitude extremes of $v$ and $w$. Next modes (with smaller real part of the eigenvalues) show enlargement of the number of extremes at all velocity components. Such correlation between the mode number and the number of the extremes is characteristic to stability problems for many flows [5] and from
Fig. 4. Amplitudes and phases of wall-normal velocity components of the optimal disturbance and the streamwise velocity component of corresponding streaky structure.

...physical point of view is related to more pronounced role of viscosity, as velocity gradient grows. The more complicated structure of the next modes leads to the necessity to enlarge the number of knots to resolve accurately the mode structure.

Amplitudes and phases of spanwise velocity components $u$ and $v$ of the optimal disturbance (its streamwise component is close to zero as it represents in fact a streamwise vortex) and the streamwise component $w$ of the streaky structures developed to $t = t_{opt}$ (its other components are close to zero) are shown in Fig. 4. Note that the dimension of the subspace of the optimal disturbances, i.e. multiplicity of the largest singular value of matrix $\exp(t_{opt} \tilde{H})$ was equal to 1 and that this singular value (23.3) was well separated from the other ones (17.7 was the next).

The disturbance development can be interpreted physically as follows. Fluid particles from the regions of small mean streamwise velocities $W(x,y)$ shift in the cross-section plane to the regions of high velocities $W(x,y)$ and vice versa preserving their streamwise momentum. Such a mixing, called lift-up effect, is accompanied by appearance of relatively large streamwise velocity disturbances $w(x,y)$ and, hence, by the growth of kinetic energy of the disturbance. This process cannot be...
unlimited in $t$, however, as the spanwise velocity component decays gradually under effect of viscosity and the lift-up virtually stops at time $t_{\text{opt}}$, the disturbance transforming to a streaky structure with predominant streamwise velocity component and kinetic energy dissipating slowly in time.

The optimal disturbances were calculated in different subspaces of dimension $l$ of most slowly decaying modes. Fig. 5 shows the development of the maximal possible amplification of the initial kinetic energy density for disturbances of class $I$ at different $l$. The amplification reaches the maximum value at each $t$ at relatively small $l = 43$ and experiences virtually no changes at larger $l$. The corresponding maximal energy density for the optimal disturbance is $E_{\text{max}} = 540$.

Although the leading eigenvalue for class $I$ is also the leading eigenvalue of the whole equation, its optimal disturbance is not optimal for the whole system (at least at studied Re and $\alpha$). Results presented in Fig. 6 show that the maximal energy growth can be provided by the optimal disturbances of different classes at different values of streamwise wave number $\alpha$. For example at $\alpha = 0.1$ the growth of disturbance of class $II$ is dominant in which case $E_{\text{max}} = 672$.

### Table 2: Critical Reynolds numbers for disturbances of different symmetries.

<table>
<thead>
<tr>
<th>$I$</th>
<th>$II$</th>
<th>$III$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recrit</td>
<td>78.3</td>
<td>127.1</td>
</tr>
</tbody>
</table>

The critical Reynolds numbers $\text{Recrit}$ were also calculated for systems of different classes separately. The results are given in Table 2. It appeared that in all cases the disturbances with $\alpha \approx 0$ manifest the growth before any other disturbances. Global $\text{Recrit} = 78.3$ is provided by the system of class $I$. Note, that for a plane Poiseuille flow $\text{Recrit} = 49.6$ [5]. That is, in the sense of the critical Reynolds number, side walls make the flow more resistant to the disturbance energy growth.

**Acknowledgements.** This work was supported by the Ministry of Education and Science of the Russian Federation, Grant N RNP.2.1.2.3370, the Russian Foundation for Basic Research (project N 07-01-00658) and Russian Academy of Sciences (project "Optimization of numerical algorithms for solving the problems of mathematical physics")

**REFERENCES**