Turbulence Generation Modeling in Edge Wakes

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Turbulence generation in edge wakes for blade and nozzle sharp edges is considered. The proposed mathematical model for turbulence generation follows from special averaged procedure on the scale near free molecule run length [1]. This turbulence generation mechanism is described by expanded system of unsteady Euler and Navier–Stokes equations with additional dispersive terms of the third order, which are results of average procedure [2]. The presence in initial equations of the third order derivative terms is demanded for additional boundary conditions (for example, the determination also acceleration values). In total this mathematical modeling allows to simulate of turbulence generation from very small scales, beginning with a free molecule run length scales.

The paper presents the theoretical and numerical analysis for a one dimensional case of turbulence generation modeling [3]. Also the typical numerical results for a two and three dimensional cases of turbulence generation in edge wakes are considered. The solution of unsteady equation systems is obtained by S.K. Godunov method [4] of a second order of accuracy. Calculated average parameters of simulated turbulence flows have the agreement with experimental data.

On microstructural dispersion. In each mass point of space one introduces characteristic microscale \( l \) of order of the free run length of molecules, which defines the spatial size of possible collective motions and interactions of molecules, which are used for averaging the parameters in the given point [1,2]. The scale \( l \), as we shall show below, allows taking into account effects of the microstructural dispersion in phenomenological models of easy moving continuous mediums. Spatial averaging of parameters in each point is carried out taking into account the finiteness of the indicated scale \( l \). First, we shall write averaging formula for the one-dimensional case. We have

\[
\bar{u}(x,t) = \frac{1}{2l} \int_{x-l}^{x+l} u(x-\xi,t) d\xi. \tag{1}
\]

By means of the differential approximation it is expressed as

\[
\bar{u}(x,t) = u(x,t) + \sum_{p=2}^{\infty} \frac{l^{2p}}{(2p+1)!} \frac{\partial^{2p+1} u}{\partial x^{2p+1}}. \tag{2}
\]

If we restrict the required function \( u(x,t) \) by condition of a small order of members containing the fourth and higher order derivatives, we obtain approximation

\[
\bar{u}(x,t) = u(x,t) + \frac{l^2}{3!} \frac{\partial^3 u}{\partial x^3} + O(l^4). \tag{3}
\]

Replacing the initial averaged in a mass point functions \( \bar{u}(x,t) \) in the initial hydrodynamic equations by their expressions such as (2) and (3) we explicitly introduce into consideration effects of the microstructural dispersion. These effects are described by the even partial derivatives of the second and higher orders. Thus the influence of dispersion is taken into account by means of the uniform characteristic scale of dispersing process \( l \), which requires an accurate physical adequate definition. In this respect we shall mark analogy with dissipative processes of viscosity and heat transfer, which also are taken into account by introducing the uniform coefficients of viscosity and heat transfer.

The averaging procedure, similar to (1), is easy to carry out for the two-dimensional and three-dimensional cases. For any function \( f(u) \) in a mass point, we can reduce accounting of the finiteness of the dispersion microscale \( l \) to the multiplication of this function by the differential operator of the following type (in one-dimensional case)
Section II

\[ N = 1 + \sum_{p=1}^{\infty} \frac{t^{2p}}{(2p+1)!} \partial_x^{2p}. \]  

(4)

**Microstructural dispersion in the Burgers model.** We consider the Burgers equation as an elementary example of taking into account the microstructural dispersion for non-linear viscous processes [5]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad \alpha = \text{const} > 0, \]  

(5)

we shall rewrite it in a divergent form

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} - \alpha \frac{\partial u}{\partial x} \right) = 0. \]  

(6)

Supposing, that equation (6) defines the averaged value \( \bar{u} \) in a mass point and using formula (4), we derive the equation

\[ \frac{\partial}{\partial t} (Nu) + \frac{\partial}{\partial x} \left( N \left( \frac{u^2}{2} - \alpha \frac{\partial u}{\partial x} \right) \right) = 0. \]  

(7)

From (7) we obtain the quasi-linear equation of the third order with accuracy of \( O(t^4 + \alpha t^3) \)

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{t^2}{6} \frac{\partial^3 u}{\partial x^3} + \frac{t^2}{6} \frac{\partial^3}{\partial x^3} \left( \frac{u^2}{2} \right) = \alpha \frac{\partial^2 u}{\partial x^2}. \]  

(8)

It is easy to see, that equation (8) is invariant regarding the Galilean transformations. Exact stationary solutions of equation (8) and their important properties are described in [3]. Numerical solution of (8) were presented in [6].

**Microstructural dispersion in the hydrodynamic model.** We shall consider system of the hydrodynamic equations in the divergent form

\[ \frac{\partial U}{\partial t} + \nabla F = 0, \quad U = \begin{pmatrix} \rho \\ m \\ \varepsilon \end{pmatrix}, \quad F = \begin{pmatrix} m \\ \Pi - \tau \end{pmatrix}. \]  

(9)

Here \( \rho \) is the density, \( m \) is the momentum vector of the unit volume, \( m = \rho q \), \( q \) is the velocity vector, \( \varepsilon \) is the specific energy of the unit volume \( \varepsilon = \rho(e + q^2/2) \), \( e \) is the specific internal energy, \( \Pi \) is the tensor of the second rank with components (in the tensor terms)

\( \Pi_{ij} = p\delta_{ij} + \rho q_i q_j \), \( p \) is the pressure, \( \tau \) is the tensor of viscous tensions with components

\[ \tau_{ij} = \mu \left( \frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{3} \nabla q \delta_{ij} \right). \]

\( E \) is the vector of stream of energy

\[ E = m(e + p)/\rho - \tau : m/\rho - k\nabla e / c_v, \]

\( \mu \) and \( k \) are coefficients of molecular viscosity and heat transfer, \( c_v \) is specific heat at constant volume.

Averaging equations (9) by small independent on time volume \( \bar{G}(t^4 \approx G) \) expanding functions \( U \) into the Taylor series and integrating we derive the corresponding differential relations

\[ \frac{\partial}{\partial t} (N \cdot U) + \nabla \cdot (N \cdot F) = 0. \]  

(10)
Neglecting members of $O(t^4 + \mu^2)$ order in (10), we obtain the first differential approximations of the average hydrodynamic equations. As an example we shall write a form of operator $N$ for the first differential approximations in the three-dimensional case

$$N = 1 + \frac{l^2}{3l^2} \frac{\partial^2}{\partial x^2} + \frac{l^2}{3l^2} \frac{\partial^2}{\partial y^2} + \frac{l^2}{3l^2} \frac{\partial^2}{\partial z^2}.$$  

For the two-dimensional case the similar differential approximations for the Navier–Stokes equations were written in [7] basing upon conditions of invariance regarding geometrical Galilean transformations.

Integrating equations (10) we can obtain and analyze behaviour of averaged solutions of the hydrodynamic equations (by analogy with explained in the previous section example of the averaged solutions of the Burgers equation).

**Critical Reynolds number and critical speed of transition.** Validity of the continuity equations describing continuous solution will be violated when the gradients of the microscopic variables become sufficiently large and defined by them the length scales are comparable with the average length of the molecule free run length. Possibility to describe such areas of the large gradients (infinite in the limit) in the Euler model for the ideal gas is provided by introducing the relations at the discontinuity, which follow as well as differential equations from the integral laws of conservation of mass, impulse and energy. Using model of the viscous heat conducting gas we represent such areas as narrow transition fronts (size of length of molecule free run length). To close equations of motion of the viscous heat conducting gas we need describe friction tension and heat flow through the microscopic parameters. Here we can use kinetic theory to calculate coefficients of friction and heat transfer. Thus value of the kinematical viscosity $\nu$ can be written with the help of the average velocity of the chaotically moving molecules $\nu_{av}$ and length of the free run of molecules $l$ as follows

$$\nu = 0.499 \nu_{av} l \approx 0.5 \nu_{av} l .$$  

We will write non-dimensional coefficient before the addendums defining viscous effects in the Navier–Stocks equations as the inverse Reynolds number

$$\frac{1}{Re} = \frac{\nu}{UL},$$  

where $U$ and $L$ are characteristic velocity and the length of the considered microscopic process. Substituting (12) into (13) we obtain

$$\frac{1}{Re} = \frac{\nu_{av} l}{2UL}.$$  

Non-dimensional value $l/L$ called the Knudsen number $Kn$. It characterizes influence of scale of effects of molecular transfer. In order to keep validity of continuous approach it is necessary that Knudsen number is small compare to 1.

We can rewrite relation (14) in the following form

$$\frac{\nu_{av}}{U} \cdot Kn \cdot Re = 2$$  

and we introduce another non-dimensional number

$$I = \frac{\nu}{l},$$  

which characterizes dynamics of molecular transfer for the used microscopic approach. Then (15) can be written in the form

$$l \cdot Kn \cdot Re = 2.$$  

Number $I$ as we shall see further is a very useful parameter when studying viscous transition processes.
Performed in [3] analysis of the steady solutions of equation (8) for the considered case allows to introduce the critical Reynolds number. When this value is approached, then for solution in shape of the transition front the type of two defining singular points changes from node to focus. This number is obtained when the expression under the root sign is equal to zero, when

$$\alpha = 2lU / \sqrt{6}.$$ 

We can rewrite this relation in form of the critical Reynolds number

$$Re_* = lU / \alpha = \frac{3}{\sqrt{2}}$$

(18)

Assuming in accordance with (12) coefficient of kinematical viscosity $\alpha = 0.5v_{av}/l$ from (18) we obtain critical speed $U_*$ and critical number $I_*$ of transition (value when character of the singular points changes)

$$U_* = \frac{3}{\sqrt{2}} \frac{\alpha}{l} = 0.61v_{av}, \quad I_* = \frac{v_{av}}{U_*} = 1.64$$

(19)

We should also underline that such values of the critical speed, critical Reynolds number and critical number $I_*$ follow from analysis when assuming validity of the Burgers equation and considered micro structural turbulence model for description of the one dimensional viscid flow.

**Numerical results.** Some results of turbulent edge wake modeling were presented in [4], where numerical solutions of the Euler equation system were obtained by the Godunov method of the second order of accuracy. Here we use the same numerical code. These calculations were fulfilled by S.A. Sherbakov.

The first results relates to flat channel flow with sharp edge. We use the experimental data [8], when $M_1 = 1.85$ in the upper side and $M_2 = 0.75$ in the bottom side, $u_2 = 0.53u_1$, $\rho_2 = 0.67\rho_1$ and $\gamma = 1.4$. The uniform grid has 1000x200 cells. The pressure, entropy and turbulent energy are presented for the time $t=100$ in fig. 1.

![Fig.1. Edge wake vortex structure as contour lines of pressure with $\Delta p = 0.012$ (a), entropy (b) and turbulence energy (c); grid 1000x200 cells.](image-url)
In this case we have unsteady solution, which doesn’t depend on initial data, beginning with time \( t \approx 20 \). We can see a turbulence generation by \( x > 5 \) and a typical distribution of the turbulence energy. Comparison of numerical results with experimental data [8] are shown in fig. 2 by \( x = 7 \).

![Fig. 2. Comparison with experimental data for average axial velocity (a) and Reynolds stress (b) by \( x = 7 \).](image)

The same results for grid 500x100 cells are demonstrated on fig. 3. For more detail grid we have more good comparison with experimental data.

![Fig. 3. Edge wake vortex structure as contour lines of pressure with \( \Delta p = 0.006 \) (a) and entropy (b); grid 500x100.](image)

Following results are related to two subsonic flow with \( M_1 = 0.25 \) (by \( y > 0 \)) and \( M_2 = 0.5 \) (by \( y < 0 \)). Initial data in \( t=0 \) are \( p_1 = 1.1 \rho_2, \rho_1 = 0.9 \rho_2 \). Vortex distribution is shown in fig. 4 for the time moment \( t=18 \) for two grids 500x400 (a) and 1000x800 (b).
The last results were obtained for the subsonic jet with \( M = 0.2 \), which propagated from a flat nozzle in the supersonic flow with \( M = 2 \). Pressure and density in jet on two times more than in supersonic flow. Pressure and entropy fields in the time moment \( t=6 \) are shown in fig. 5 (grid 1350x400 cells).

The presented results have demonstrated the possibility of turbulence generation modelling with help of the additional dispersive terms of the third order, which are results of average procedure.
References