ON SOUND PROPAGATION IN ATMOSPHERE IN THE SHALLOW WATER MODEL

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1. Introduction. Sound characteristics for the model of shallow water on a rotating attracting sphere for some simple solutions are studied [1,2].

The shallow water model is described by a hyperbolic system due to a gas-dynamic analogy. Its solutions may be treated as gas motions on the surface of a rotating sphere. The layer depth \( h \) is identified with density, and the equation of state has the form \( p = \frac{f_0}{2} h^2 (h \equiv \rho) \), \( f_0 \) is the Froude number equivalent to entropy.

The system of equations is of the form

\[
Dv = w^2 \cot \theta + r_0 w \cos \theta + \frac{1}{4} r_0^2 \sin \theta \cos \theta - f_0 h \theta, \\
Dw = -v w \cot \theta - r_0 w \cos \theta - f_0 (\sin \theta)^{-1} h \theta, \\
Dh = (\sin \theta)^{-1} h \left( w_{\phi} + (v \sin \theta)_{\theta} \right) = 0,
\]

where \( D = \partial_t + v \partial_\theta + (\sin \theta)^{-1} w \partial_\phi \). Equations (1.1) are recorded in spherical coordinates; \( 0 < \theta < \pi \) is a latitude, \( 0 \leq \phi < 2\pi \) is a longitude; \( v \) is the meridional component of velocity and \( w \) is the longitudinal one. Dimensionless parameters \( r_0 \) and \( f_0 \) are connected with the numbers of Rossby \( R_0 \) and Froude \( F \) by relations \( r_0 = R_0^{-1}, f_0 = F^{-2} \). In gas-dynamic treatment of system (1.1) \( r_0 \) is also equivalent to the Rossby number.

2. Equations of sound characteristics. Sound velocity is determined by formula \( c = \sqrt{f_0 h} \).

Let a family of sound characteristics be specified by equations \( \chi(t,r,\theta,\phi) = \text{const} \). Then, for the given solution \( u = (u,v,w), h \) system (1.1) \( (u = 0 \) - is a radial component of velocity) the function \( \chi \) satisfies the equation

\[
\chi_t + r^{-1} v \chi_\theta + (r \sin \theta)^{-1} w \chi_\phi = \varepsilon c N, \quad (\varepsilon = \pm 1),
\]

where

\[
N = |\nabla \chi| = \left( \chi_r^2 + r^{-2} \chi_\theta^2 + (r \sin \theta)^{-2} \chi_\phi^2 \right)^{1/2}.
\]

Sound perturbations propagate both over the sphere surface and in the radial direction although the gas motion itself occurs over the sphere surface.

Equation (2.1) is the Hamilton-Jacobi equation for the function \( \chi = \chi(t,x_1,\ldots,x_n) \) and it is recorded in the form

\[
\chi_t + H(t,x_1,\ldots,x_n,\chi_1,\ldots,\chi_n) = 0,
\]

where \( \chi_i = \partial \chi / \partial x_i, i = 1,\ldots,n \). Characteristic system for Eq. (2.3) is called the canonical system of differential equations, has the form

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\[
\frac{dx_i}{dt} = H_{x_i}, \quad \frac{d\chi_i}{dt} = -H_{\chi_i}, \quad (i = 1, \ldots, n) \tag{2.4}
\]

and is the Hamiltonian system of equations with the Hamiltonian \(H\) [3].

Let us consider sound characteristics and bicharacteristic of Eq. (1.1) using some simple solutions having great physical importance: the state of equilibrium and shear zonal motions.

Firstly, let us note the following fact. As the gas motion described by Eqs. (1.1) does not depend on \(r = \sqrt{x^2 + y^2 + z^2}\), one can consider characteristics that do not depend on radius either, i.e., solutions of equations (2.1) (and systems (2.4)) independent of \(r\). They will describe traces on the sphere of sound surfaces propagating in space. Let us call the investigation problem of such characteristics a restricted problem (we mean a sphere restriction).

3. Sound characteristics on the equilibrium state. Equations (1.1) admit the equilibrium state, in which relative components of velocity \(v = w = 0\), depth profile (density) has the form

\[
h = \alpha_0^2 (k_0^2 + \sin^2 \theta), \tag{3.1}
\]

where \(\alpha_0^2 = r_0^2 / 8 f_0, k_0^2 = 8 f_0 h_0 / r_0^2, h_0 > 0\) are constants. The sound velocity in this solution is 
\[c = (r_0 / 2\sqrt{2})(k_0^2 + \sin^2 \theta)^{1/2}.\]
The equation

\[
r = \alpha_0^2 (k_0^2 + \sin^2 \theta) \tag{3.2}
\]

for \(\theta \in (0, \pi)\) specifies equilibrium rotation surface (isochoric surface) in space \(R^3(x)\).

Since equation (2.1) is homogeneous with respect to derivatives of the function \(\chi\), due to multiplying it by a constant factor the Hamiltonian \(H\) for Eq. (2.1) in this solution may become

\[
H = (k_0^2 + \sin^2 \theta)^{1/2} \left( \chi_r^2 + r^{-2} \chi_\theta^2 + (r \sin \theta)^{-2} \chi_\phi^2 \right)^{1/2}. \tag{3.3}
\]

We denote

\[
Q = (k_0^2 + \sin^2 \theta)^{1/2}, \tag{3.4}
\]

then \(H = QN\).

Hamiltonian system (2.1) for Hamiltonian (3.3) is of the form \((x_1, x_2, x_3) = (r, \theta, \phi)\)

\[
\begin{aligned}
\frac{dr}{dt} &= \frac{Q}{N} \chi_r, \\
\frac{d\theta}{dt} &= \frac{Q}{N} \frac{\chi_\theta}{r^2}, \\
\frac{d\phi}{dt} &= \frac{Q}{N} \frac{\chi_\phi}{(r \sin \theta)^2}, \\
\frac{d\chi_r}{dt} &= \frac{Q}{r^2 N} \left( \chi_r^2 + (\sin \theta)^{-2} \chi_\phi^2 \right), \\
\frac{d\chi_\theta}{dt} &= 0, \\
\frac{d\chi_\phi}{dt} &= -\frac{\sin \theta \cos \theta}{QN} \left( \chi_r^2 + r^{-2} \chi_\theta^2 - k_0^2 (r \sin \theta)^{-2} \chi_\phi^2 \right).
\end{aligned} \tag{3.5}
\]

As \(H_r = H_\phi = 0\), Hamilton system (3.5) has integrals \(H = k_0, H_\phi = l_0\), where \(k_0, l_0 = \text{const}\). It is too complicated for analytical study; its solutions are found numerically.

Let us consider the restricted problem for the given solution: investigating characteristics \(H(t, \theta, \phi) = \text{const}\) on the sphere surface. Then, the Hamiltonian of the problem equals

\[
H_1 = QN_1, \tag{3.6}
\]

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where

\[ N_1 = \left( \chi_\theta^2 + \sin^2 \theta \right)^{1/2}. \]  

(3.7)

Canonical system (2.4) for Hamiltonian (3.6) is recorded as follows:

\[
\frac{d\theta}{dt} = \frac{Q}{N_1} \chi_\theta, \quad \frac{d\phi}{dt} = \frac{Q}{N_1} \chi_\phi, \\
\frac{d\chi_\theta}{dt} = \frac{\sin \theta \cos \theta}{QN_1} \left( \chi_\phi^2 - k_0^2 \sin^4 \theta \chi_\phi^2 \right), \quad \frac{d\chi_\phi}{dt} = 0.
\]  

(3.8)

System (3.8) is integrated in elliptic functions but its solution is very tedious. We describe one of ways of integrating this system.

Let us introduce a function \( \psi = \chi_\theta / \chi_\phi \). Then, the combination of the third and the fourth equations of system (3.8) gives

\[
\frac{d\psi}{dt} = -\psi^2 \sin \theta \cos \theta + k_0^2 \csc \theta \left( \psi^2 + \sin^2 \theta \psi^2 + \csc^2 \theta \right).
\]  

(3.9)

We introduce a function \( \tau = \tau(\phi) \) such that

\[ \tau(\phi) = \cot \theta(\phi). \]  

(3.10)

Then, system (3.8), (3.9) takes the form

\[
\frac{d\tau}{d\phi} + \psi = 0, \\
\frac{d\psi}{d\phi} = \frac{\tau(k_0^2(1+\tau^2) - \psi^2)}{(1+\tau^2)(1+k_0^2(1+\tau^2))}.
\]  

(3.11)

After substituting the function \( \psi \) from the first equation into the second equation of system (3.11) and introducing a new function \( \mu = \mu(\tau) \) such that

\[ \mu(\tau) = \frac{d\tau}{d\phi} \]  

(3.12)

system (3.11) is reduced to a linear inhomogeneous equation with respect to a function \( \sigma = \mu^2 \):

\[
\frac{d\sigma}{d\tau} = \frac{2\tau}{(1+\tau^2)(1+k_0^2(1+\tau^2))} \sigma + \frac{k_0^2 \tau(1+\tau^2)}{1+k_0^2(1+\tau^2)} = 0,
\]  

(3.13)

the general solution of which is recorded in the form of tedious combination of elliptic integrals.

Homogeneous linear Eq. (3.13) has a general solution

\[ \sigma = \sigma_0 \left( \frac{1+\tau^2}{a_0^2 + \tau^2} \right), \quad \sigma_0 = \text{const}, \]

where \( a_0^2 = 1+k_0^2 \). If we vary the constant \( \sigma_0 \), we find the general solution of inhomogeneous Eq. (3.13) in the form...
\[ \sigma = \frac{l_0(1 + \tau^2) - \tau^2(1 + \tau^2)}{2(a_0^2 + \tau^2)}, \quad l_0 = \text{const}. \]

Then, the solution of system (3.11) is presented by a formula

\[ \phi - \phi_0 = \int \frac{d\tau}{\mu(\tau)}, \]

where

\[ \mu(\tau) = \left[ \frac{-\tau^4 + (l_0 - 1)\tau^2 + l_0}{2(a_0^3 + \tau^2)} \right]^{1/2}, \]

This is a combination of elliptic integrals of different kinds.

The numerical analysis gives well-treated results.

4. Numerical analysis of a characteristic conoid for Restricted problem. The behavior of characteristics depends on parameter \( k_0^2 \) that characterizes rotation velocity of the sphere and \( \theta_0 \) determining the position of the source of sound perturbations with respect to latitude.

It is easy to control the influence of these parameters if we observe the conoid at enough much time of perturbation propagation, not long before the moment when caustics appear on the perturbation front.

4.1 Influence of \( k_0 \)

If we fix the liquid depth and the sphere radius. Then, minor rotation velocities correspond to major values of \( k_0 \). As \( k_0 \to \infty \), the rotation velocity tends to zero.

For simplicity we assume that the conoid vertex is at the equator.

- When the rotation is absent \( (k_0 \to \infty) \), the perturbation front is always a circumference and it all converges to the point diametrically opposite to the conoid vertex.

- When \( k_0 \) decreases, the velocity of perturbation propagation decreases as well, and the perturbation front is extended in the latitudinal direction. The following fact is of much interest: as \( k_0 > 0.37 \) caustics are formed on the front nearer to the poles, and for minor \( k_0 \) they are formed at the equator.
For $k_0 = 0$ and $k_0 = \infty$ equations for bicharacteristics are integrated in elementary functions that, apparently, makes it possible to substantiate just such behavior of the conoid.

4.2 **Influence of $\theta_0$**

- When the conoid vertex is at the equator, the conoid is symmetric with respect to it.
- If the conoid vertex is at southern hemisphere, the perturbation front converges to northern hemisphere.

Such behavior of bicharacteristics is natural due to the fact that the perturbation propagation velocity is independent of the longitude $\phi$.

5. **Sound characteristics on zonal flow.** Equations (1.1) have the solution

\[ v = 0, \quad w = w_0 \sin \theta, \quad h = \beta_0^2 (k_0^2 + \sin^2 \theta), \]  

(5.1)

where

\[ \beta_0^2 = \frac{1}{2f_0} \left( w_0 + \frac{r_0}{2} \right)^2, \quad k_0^2 = \frac{k_0}{\beta_0^2}, \]  

(5.2)
such that \( c = \frac{1}{\sqrt{2}} \left( w + \frac{r_0}{2} \right) \left( k_0^2 + \sin^2 \theta \right)^{1/2} \). As \( w_0 = 0 \), solution (5.1) passes to a relative rest, the state of equilibrium, considered in section 3. Equation (2.1) on solution (5.1) is of the form

\[
\chi_t + r^{-1} w_0 \chi_{\phi} - \varepsilon (k_0^2 + \sin^2 \theta)^{1/2} \left( \chi_r^2 + r^{-2} \chi_{\theta}^2 + (r \sin \theta)^{-2} \chi_{\phi}^2 \right)^{1/2} = 0,
\]

(5.3)

where \( \alpha_0 = \frac{1}{\sqrt{2}} \left( w_0 + \frac{r_0}{2} \right) \). The Hamiltonian for Eq. (5.3) is specified by a formula

\[
H = a_0 r^{-1} \chi_{\phi} - \varepsilon (k_0^2 + \sin^2 \theta)^{1/2} \left( \chi_r^2 + r^{-2} \chi_{\theta}^2 + (r \sin \theta)^{-2} \chi_{\phi}^2 \right)^{1/2},
\]

(5.4)

differing from (3.3) by the first term. It makes a contribution to the equation for \( \frac{d \chi_r}{dt} \) and \( \frac{d \phi}{dt} \).

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