Modification of Feldkamp algorithm for bifocal tomography

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ABSTRACT

The present work examines the Feldkamp reconstruction algorithm as applied to tomography reconstruction of the internal structure of a three-dimensional object from projection data measured in a bifocal geometry. To make the reconstruction from such data possible, the backprojection procedure in this algorithm was substantially modified. Considerable attention was focused on filtration realization in the algorithm. Numerical simulations have been carried out.

KEY WORDS: modeling, tomography, bifocal geometry, Feldkamp algorithm

1. INTRODUCTION

Tomographic methods for reconstructing the internal structure of various objects are widely used in medicine, in some branches of industry, and also in doing physical research. The tomography fundamentals, including the main physical and mathematical concepts underlying them, are clearly outlined, for instance, in the monograph [1]. A more strict description of its mathematical tools is given in [2].

Usually (and also here), a ray tomography approximation is dealt with. This approximation implies that, to each measurement \( f_\alpha \), a line integral of a sought function \( g(x,y,z) \) along a certain straight line corresponds. The ray approximation may be realized using various hardware configurations, the particular type of which depends on the properties of the object under study and the measurement facilities used. For the sake of definiteness, we consider the problem of probing a solid body with penetrating radiation or particles. The conditions for the ray approximation are fulfilled, in particular, if the detectors are well collimated and absorption prevails over other types of interaction of the probing beam with the substance. Here, we assume that the data are measured using plane imagers, which justifies using the term two-dimensional projections. If an object under study is illuminated by a source that produces a wide diverging beam, then the distribution of the transmitted intensity over the detecting matrix forms a cone-beam two-dimensional projection (see below). Such a measurement scheme is often used in medical and industrial tomography equipment. In the present work, a generalization of the cone-beam case is introduced, termed bifocal projection geometry. This geometry arises when a source emits a narrow beam swept in two orthogonal planes, in different focal points. In this case, the detector measures a two-dimensional bifocal projection. In our opinion, the bifocal data acquisition system offers much more promise in many applications since it is more flexible compared to the conical case. In particular, it may be realized in proton medical tomography, where a patient body is probed with a proton beam.

The aim of the present work is to examine one of the approximate reconstruction algorithms, known in the literature as the Feldkamp algorithm. This algorithm was developed in [3] to reconstruct a function of three variables from a set of its cone-beam two-dimensional equatorial projections. In the present study, we consider a possibility of using this algorithm to a case in which a data set measured in a bifocal geometry is available. In this connection, several modifications have been introduced into the algorithm. Additionally, considerable attention has been paid to realization of filtration, which constitutes part of the integral reconstruction procedure. Numerical simulations have been carried out.

2. CONE-BEAM AND BIFOCAL PROJECTION-DATA ACQUISITION SCHEMES

2.1. Conical geometry

Let a source of a penetrating radiation (or particles) be located at a point \( S \). This source illuminates, with a widely diverging beam, an object under study. On the other side of the object, in a plane \( D \), a two-dimensional imager (detecting matrix) is installed. The rotation of the source-detector system relative to the object allows exposure of the latter with the generated beam under various illumination angles. The distribution of the intensity registered by the detector is called a cone-beam projection. In the case of the ray transmission tomography, an equation for a cone-beam projection with a source located at a point \( S \) is
2.2. Bifocal geometry

Bifocal projection-data acquisition geometry is illustrated in Fig. 1. A source of radiation or charged particles, D – a detector matrix, the source-to-origin distance is \( r_s \), the origin-to-detector distance is \( r_o \), the beam sweeps the object being swept in the horizontal direction at the points of the straight line \( l \).

2.3. Feldkamp algorithm for a bifocal projection geometry

In the limit of an infinitely large number of measurements, the tomography reconstruction in the ray approximation reduces to inverting \( P \) - or \( D \)-transforms, which are well known in integral geometry [2]. A great number of algorithms for such reconstruction have been developed, based on the inversion formula for these transforms [4,5,6,7]. The algorithms that realized the exact formulas require measurement from a great number of directions for achieving a satisfactory reconstruction quality, which is often impossible in actual cases. In such situations, algorithms may be used which just approximate the inversion equations. One of such algorithms is the Feldkamp algorithm that may be referred to backprojection filtration algorithms. This algorithm was first proposed in [3] for the case of a conical equatorial geometry. According to [3], the Feldkamp algorithm may be formulated as follows:

\[
g(x,y,z) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{(r_s + r_o)^2}{(r_s + r_o + x\cos\theta + y\sin\theta)^2} \times \int_0^{2\pi} Y(x,y,z,\omega) \left( x' - x \right) dx' + \cdots \]

where

\[
Y(x',y',z) = \frac{r_s + r_o}{r_s + x\cos\theta + y\sin\theta}, \quad z = \frac{r_s + r_o}{r_s + x\cos\theta + y\sin\theta},
\]

\[
f_s(q) = \ln \left( \frac{I_o}{I(q)} \right) = \int_u g(S + pn) \, dp,
\]

\[
q = (r_s + r_o) \left( \frac{S}{r_s} - \frac{n}{\cos(S,n)} \right)
\]

In Equations (2), \( v \) is a basis vector along the V-axis at the detector, the point \( S_1 \) is shown in Fig. 1, and all other notations are the same as in (1).
Here $\phi$ is the azimuthal angle; and $\omega_{r,0}$ and $\omega_{z,0}$ in (6) are the cut-off frequencies. For $r_s \to \infty$ algorithms (3)-(6) go into the exact slice-by-slice inversion formula for the parallel-beam limiting case. Besides, equations (3)-(6) give the exact result in the plane $z=0$ for any distance to the source.

To accommodate the Feldkamp reconstruction algorithm to the case of a bifocal projection geometry, we notice that the case under consideration may be considered as the one resulting from superposition of two fan-beam data acquisition schemes in two orthogonal planes. In this case, the origin-to-source distances are $r_s$ and $(r_s-d_f)$, respectively. It is seen from formula (5) that the convolution of the two-dimensional projection with the kernel is performed independently along each variable. This makes it possible to represent the Feldkamp algorithm for a bifocal geometry in the form

$$g(x, y, z) = \frac{1}{4\pi^2} \int \left(\frac{r_s + r_s - d_f}{r_s + r_s - d_f + x \cos \phi + y \sin \phi}\right) \times \tilde{f}_0 \left[Y(x, y, z), Z(x, y, z)\right] d\phi,$$

(7)

where

$$Y(x, y, z) = \left(\frac{r_s + r_s - d_f}{r_s + r_s - d_f + x \cos \phi + y \sin \phi}\right),$$

$$Z(x, y, z) = \left(\frac{r_s + r_s}{r_s + x \cos \phi + y \sin \phi}\right).$$

(8)

To get a more accurate result, one should take into account the dependence on the parameter $\omega_{r,0}$. The influence of this parameter on reconstruction quality was examined in the course of the numerical simulations. Here again, the convolution along the horizontal coordinate with the function $g_r$ from (6) was included. Here, there arises a problem that concerns the value of the parameter $\omega_{r,0}$. The influence of this parameter on reconstruction quality was examined in the course of the numerical simulations. Here again, the convolution along the horizontal coordinate with the function $g_r$, as in the previous method, was realized as a Shepp-Logan filtration procedure.

It has been shown in [3] that the double convolution (5) with $g_r$ and $g_z$ from (6) may be replaced by the following expression:

$$\tilde{f}_0 (Y, Z) = \Re \int \hat{f}_0 \left(\hat{g}_r (Y'), \hat{g}_z (Z')\right) \times$$

$$\times \int \frac{g_r (Y-y') f_r (Y', Z')}{\sqrt{1 + \left(\frac{Y'}{r_s + r_s - d_f}\right)^2 + \left(\frac{Z'}{r_s + r_s}\right)^2}} dY' dZ' \times$$

(9)

The functions $g_r (Y)$ and $g_z (Z)$ are given, as before, by expressions (6).

4. REALIZATION OF FILTRATION IN THE FELDKAMP ALGORITHM

Equation (3), as well as (7), describes the backprojection procedure in the case of filtered two-dimensional projections $\tilde{f}_0 (Y, Z)$ taken in an equatorial geometry. Filtration is made through convolution with the functions $g_r (Y)$ and $g_z (Z)$. In the present study, three realizations of the projection filtration in the Feldkamp reconstruction algorithm were considered. The first realization is a simplest and most quick one. As mentioned previously, equations (3)-(6) give an exact solution for the case in which the source is located infinitely far from the object. In this case, the three-dimensional problem separates into two-dimensional problems in the planes orthogonal to the Z-axis. Formally, this situation corresponds to the case $\omega_{z,0} \to \infty$, when $g_z$ being just the sinc function, transforms into a Dirac delta-function $\delta$. Thus, for sufficiently large distances $r_s$ ($r_s-d_f$ in the case of a bifocal geometry) the convolution with $g_z$ may be omitted. Then, the convolution with the function $g_r$ has to ensure reconstruction in a plane.

In the present study, this convolution was realized as the well-known Shepp-Logan filtration procedure [8]. This algorithmic realization of equations (5) or (9) with $g_r$ and $g_z$ from (6) is largely an approximate one (it should be noted here that the Feldkamp algorithm itself is an approximate method) since the dependence on the vertical coordinate $z$ in it is taken into account only through the expression in the denominator. An approximation obtained is the more accurate, the farther the source is located from the object.

To get a more accurate result, one should take into account the dependence on $z$ through the convolution with the sinc function. In this connection, into the second filtration method used in the present study, immediate computation of the convolution of a projection along the vertical coordinate with the kernel $g_z$ from (6) was included. Here, there arises a problem that concerns the value of the parameter $\omega_{z,0}$. The influence of this parameter on reconstruction quality was examined in the course of the numerical simulations. Here again, the convolution along the horizontal coordinate with $g_r$, as in the previous method, was realized as a Shepp-Logan filtration procedure.

In equation (10), $Y(x, y, z)$ and $Z(x, y, z)$ are given by formulas (4). Formula (10), properly modified for the case of a bifocal geometry, was used as a basic one for the third algorithmic realization of the filtration projection procedure in the Feldkamp algorithm.

In the next section, in which the numerical simulations results obtained are described, the above-indicated realizations of filtration (9) will be referred to as the first, second, and third realizations.

5. NUMERICAL SIMULATIONS

5.1. Mathematical model
The applicability of the Feldkamp reconstruction algorithm for a three-dimensional tomography reconstruction in the case of bifocal projection data acquisition geometry was tested in the numerical simulations. The three-dimensional mathematical phantom was chosen as a cylinder of height 1.6, base radius 0.1 and density 0.5, with six attached balls of radius (density) 0.16 (1.0), 0.16(0.8), 2×0.13(0.9), 0.095(0.8), and 0.095(1.0). The model was calculated on a cubic 129×129×129 grid set in a cube with the center at the origin and the cube side equal to two conventional units. The reconstruction was performed on the same grid. The model used is depicted in Fig.2.

Fig.2: The three-dimensional phantom.

Since the Feldkamp algorithm is intended for performing reconstruction from data taken in the equatorial measurement scheme, the projection data were modeled as two-dimensional equatorial projections equally spaced along the angle ϕ in the range from 0° to 360°. In each of these projections, a 129×129 grid was used. Both noiseless data and data superimposed with random noise were used as an initial data set.

To examine the reconstruction accuracy, we calculated the normalized mean-root-square error $\Delta$. The value of $\Delta$ was calculated from the formula

$$\Delta = \frac{\|g_M - g\|}{\|g_M\|}.$$  \hspace{1cm} (11)

In (11), $g_M$ is the exact model, $g$ is the reconstructed solution, and $\|\|$ is the Euclidean norm in a finite-dimensional vector space.

5.2. Effects of the distance to the source and distance between the foci

In this section, we examined the effect of geometric parameters $r_s$ and $d_f$, which characterized the location of focal points, on reconstruction quality. The total number of available equatorial projections $M$ was equal to 60.

Figure 3 shows the reconstruction error $\Delta$ versus the source-to-origin distance $r_s$ for the fixed distance between the foci equal to two conventional units. Curves 1, 2, and 3 refer to the first, second, and third realizations of filtration (9), respectively (see section 4).

Figure 4 shows the reconstruction error $\Delta$ as a function of the distance $d_f$ between the foci. Here, the source-to-origin distance $r_s$ equaled eight conventional units. As in Fig.3, curves 1, 2 and 3 refer to different realizations of filtration (9).

It is seen from Figs. 3 and 4 that in many cases the adopted filtration procedures yield almost the same results. Nevertheless, it may be concluded that the third realization of the filtration (by formula (10)) always ensures a smaller reconstruction error. For the second realization, the computations showed that the reconstruction error $\Delta$ as a function of $\omega_0$ exhibits a flat minimum that gets displaced towards larger
values of $\omega_{z0}$ as the distance from the focal points to the studied object increases. In fact, $\omega_{z0}$ here is a regularization parameter, which, as was noted in [9], may play a part also in reconstruction from exact (i.e., noiseless) data sets.

The reconstruction quality can be visually estimated inspecting Fig.5. This figure shows the 3D image of the model object reconstructed from 60 two-dimensional projections for $r_s=8$ and $d_f=2$.

Fig.5: Reconstructed 3D image; the third filtration procedure; $r_s=8$, $d_f=2$, $M=60$, and $\Delta=0.301$.

5.3. Dependence on the total number of available projections

A series of computations was carried out to elucidate the reconstruction quality ensured by the Feldkamp algorithm on the total number $M$ of available two-dimensional projections.

Fig.6: The reconstruction error $\Delta$ versus the number of available projections $M$ ($r_s=8$, $d_f=2$); curves 1, 2, and 3 refer to the first, second, and third realization of filtration (9).

Figure 6 depicts the reconstruction error $\Delta$ as a function of the total number of available two-dimensional equatorial projections. The origin-to-source distance, $r_s$, equaled eight conventional units, and the distance between the foci, $d_f$, equaled two such units. As previously, curves 1, 2, and 3 refer to the above-described realizations of filtration (9). It is seen from the figure that the Feldkamp reconstruction algorithm displays a saturation with respect to the total number of viewing directions, i.e., for the values of $M$ larger than 100 (for the studied model), the reconstruction error $\Delta$ only weakly depended on $M$. Such a behavior is typical of many tomography reconstruction algorithms and, in particular, of algebraic reconstruction methods.

5.4. Random noise in projection data

The stability of a tomographic algorithm against a random noise is one of its most important characteristics. In this connection, a number of reconstructions were obtained from model data distorted by a random-noise. The noise was assumed to be a Gaussian one with zero average value and a variable dispersion that amounted to $\xi$ percents of the projection value at a given point.

Fig.7: The reconstruction error $\Delta$ versus the noise amplitude $\xi$ ($r_s=8$, $d_f=2$, $M=60$); curves 1, 2, and 3 refer to the first, second, and third realization of filtration (9).

Figure 7 shows the reconstruction error as a function of the noise level. A total of 60 bifocal projections were considered, the source-to-origin distance was eight units, and the distance between the foci equaled two units. The curve numeration here is the same as before. It is seen from the figure that the first realization of filtration turns out to be unstable against the noise. In contrast, for the third realization the reconstruction quality increases only insignificantly even in the case of a 20% noise. The latter finding may be explained by the fact that, as was mentioned above, $\omega_{z0}$ presents a regularization parameter, and one can suppress the noise in the third realization decreasing the value of $\omega_{z0}$. In this case, to find the value of $\omega_{z0}$, one should invoke general methods for solving ill-posed problems [10]. For the two other filtrations, there is no such a built-in regularization parameter; for this reason an additional regularization procedure should be additionally included in these filtrations. In this case, it may be
expected that curves 1 and 2 in Fig.7 should approach curve 3.

6. CONCLUSION

We have shown that the Feldkamp reconstruction algorithm with a properly modified backprojection procedure may be used for tomographic reconstruction from a measured data set taken in a bifocal-geometry measurement scheme.

Three realizations of two-dimensional projection filtration have been considered, including a simplified one, a slice-by-slice Shepp-Logan filtration procedure. The three realizations were compared in the course of numerical simulations. It was obtained that in many cases it is quite possible to use the simplified filtration without any substantial deterioration of reconstruction quality.

A comparison of the Feldkamp algorithm with other reconstruction methods, in particular, ART, has been also made. Both algorithms were shown to provide almost identical reconstruction qualities, but the computing time required for the whole reconstruction by the Feldcamp algorithm to be performed was nearly the same as the time required for making one ART iteration.

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