POD ANALYSIS OF SHOCK WAVE/TURBULENT BOUNDARY LAYER INTERACTION

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We performed a POD (proper orthogonal decomposition) analysis on a DNS result of a Ma=2.25 oblique shock wave turbulence boundary layer interaction in this paper. POD is a model reduction technique that can be used to get an optimal low dimensional representation of a high dimensional data set. Low dimensional coherent structures which contain most of the energy can be extracted by applying POD to a full information DNS result or experiment data.

We found that first few modes contain most turbulent energy of the flow. Time history of POD coefficient shows that the most energetic modes tend to have lower frequencies, which may be attributed to low frequency unsteadiness of shock motion and large scale coherent structures of turbulent boundary layer. Flowfields of primary POD modes manifested streamwise vortices before interaction and “bump-like” structures in the interaction region.

1. Introduction

Shock wave and turbulent boundary layer interaction (SWBLI) occurs ubiquitously in supersonic flows and have great impacts on the performance of high-speed vehicles. Substantial efforts have gone into this field in the past fifty years due to its vital importance in both scientific research and industry. However, some mechanisms are yet to be discovered. The development of this subject have can be seen in the review by Dolling et al.[1]

In the study of turbulence and aerodynamics, numerical simulation is playing a more and more important role with the booming of computing power. Direct numerical simulation (DNS) is becoming a common tool of turbulence research community. Using DNS, the dissipation scale of the flow is resolved and full information of the turbulence field can be obtained. However, the result of DNS calculation is comprised of huge amount of data which contains the time-space evolution of turbulence flowfield. Mining and extracting the flow mechanisms behind the data is even more important than the DNS calculation itself.

The idea of “coherent structures” developed many years ago and was firstly observed by Roshko et al.[2] in plane mixing layers experiment. Recently discovered large-scale motions (LSMs) and very large-scale motions (VLSMs)[3] in wall-bounded shear flows are also thought to be closely related with the coherent structures. The finding that turbulent flow is not purely chaotic but embodies orderly structures has a profound effect on fluid mechanics. The study of coherent structures will greatly facilitate our understanding of the turbulence dynamics.

One method of extracting coherent structures from flowfield data is POD. Proper orthogonal decomposition (POD) is a widely used technique to get an optimal low dimensional estimation of a high dimensional dataset. It has many applications in different areas. As in fluid dynamics, POD extracts modes that are linearly combination of each flowfield snapshot, which contain most of the turbulent energy.

A POD analysis of SWBLI DNS result is carried out in this paper in order to understand the coherent structure dynamics in such a case. The configuration of SWBLI studied is a flat plate turbulent boundary layer with an incident shock. Numerical issues of the DNS and POD technique is briefly reviewed, and then followed the detailed analysis of the results.
2. Numerical setting of DNS

The SWBLI studied in present paper is a supersonic turbulent boundary layer flow at Mach=2.25 with 33.2° impinging angle shock. Reynolds number is 63500/inch.

Seventh-order low-dissipation monotonicity-preserving (MP7-LD)\cite{4} scheme is adopted for the discretization of convection term. The Ducros sensor\cite{5} is used to detect shock-waves, and the MP limiter of Suresh and Huynh\cite{6} is applied to preserve the monotonicity near discontinuities. The diffusion terms of N-S equations are solved with the sixth-order compact scheme. After all the spatial terms are solved, the third-order TVD Runge–Kutta method is used for the time integration. The geometry matrix of the grid transformation is also calculated with the sixth-order compact central scheme in the conservation form to preserve the accuracy of the solution. Details of the numerical schemes used and other settings can be found in the paper of Fang et al\cite{4}.

3. POD method

POD is a widely used technique to find an optimal low-dimensional estimation of a high-dimensional dataset\cite{7}. It is applied in many different scientific research areas, including image processing, data mining, and control theory\cite{8}. In fluid dynamics, the POD procedure identifies the most energetic contributions and obtains the spatial structure of the corresponding modes. The basic idea is to describe a given statistical ensemble with the minimum number of deterministic modes. Lumley et al\cite{9} firstly used it to extract coherent structures in turbulence. There are two types of POD, the classical and the snapshot POD\cite{10}. We used the latter here because it needs much less memory than the first one in our case and a brief description in given below.

Let $q_k(x),\ k = 1, 2 \ldots N_t,$ be a set of observations (called “snapshots”) at points $x \in \Omega$, that could be obtained by a numerical simulation or experimental measurements. The goal of POD is to find functions such that

$$\frac{\langle (q, \phi)^2 \rangle}{\left|\phi\right|^2}$$

is maximized. $\langle \cdot \rangle$ is some kind of average, $\langle \cdot, \cdot \rangle$ is an inner product and $\left|\cdot\right|$ the induced norm.

The functions $\phi$ span a subspace of the original space of the snapshots $q_k(x)$ such that the error of the orthogonal projection is minimized. Solving the optimization problem leads to an eigenvalue problem where the functions are the eigenfunctions. Here we suppose that the POD modes are linear combinations of the snapshots

$$\phi(x) = \sum_{k=1}^{N_t} a_k q_k(x) \quad (1)$$

Substitute the above expression to the optimization problem would leads to the task of solving the eigenvalue problem

$$CA = \lambda A \quad (2)$$

where

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_t} \end{pmatrix} \quad \text{and} \quad c_{ij} = \frac{1}{N_t} \langle q_i(x), q_j(x) \rangle$$

$C$ is called the correlation matrix of $N_t \times N_t$ elements.
The construction of correlation matrix \( C \) makes sure that it has \( N_t \) positive eigenvalues, suppose that the eigenvalues are sorted in descending order as \( \lambda^1 > \lambda^2 > \cdots > \lambda^N_t > 0 \), and the corresponding eigenvectors \( A^i \) comprises an orthogonal set. Then \( \phi^i(x) \) is calculated as follows

\[
\phi^i(x) = \frac{1}{\sqrt{N_t} \lambda^i} \sum_{k=1}^{N_t} a^i_k q_k(x)
\]

(3)

The “energy” corresponding to a function \( \phi^i(x) \) is given by \( \lambda^i \), but what exactly this “energy” is depends on the definition of the inner product used in Eq

The functions \( \phi^i(x) \) will be called spatial eigenvectors or POD modes and \( A^i \) will be called temporal eigenvectors as the different snapshots \( q_k(x) \) might be snapshots at different points in time \( t_k \) such that \( q_k(x) = q(x,t_k) \). So the components \( a^i_k \) of the eigenvector \( A^i \) are the temporal coefficients \( a^i(t_k) \) for the spatial eigenvectors \( \phi^i(x) \)

\[
q(x,t) \approx \sum_{i=1}^{M} a^i(t) \phi^i(x)
\]

(4)

4. Results and Discussion

Firstly, we give some results of the DNS calculation and verify them with classical results. Fig. 1 is the mean flow velocity profile at \( x = 9.5 \), we can see that the profile agrees well with the wall log law, Fig. 2 is the fluctuating velocity intensity, it also agrees well with previous simulations and experiments. The good agreement of both mean and two-order statistics ensures that the present DNS can accurately simulate the turbulent fluctuating motion. The results also indicate that the flow was already fully developed turbulent boundary layer before the interaction occurs.

Fig. 1. Mean velocity profile at \( x = 9.5 \)

Fig. 2. Fluctuating velocity intensity at \( x = 9.5 \)

Fig. 3 is the shadowgraph of instantaneous \( u \) velocity at slice \( y^+=13 \). There are clearly low-speed streaks before the interaction, which is the typical characteristic of wall turbulence. The nonlinear interaction between low-speed streaks, streamwise vortices and flow instability makes the wall turbulence a self-sustaining process. After the interaction occurs, the streamwise streaks break up and its influence extends to more than ten boundary layer thickness until the typical wall turbulence is recovered.

Fig. 4 is the coherent structure identified by Q criterion with the shock wave surface, before interaction, we can observe the typical “\( \Lambda \)-shape” streamwise vortices, after the interaction, the vortices are break into finer structures and the vortices’ direction tend to be more chaotic. We can also see that the surface of the shock wave is deformed by the turbulent fluctuation.
After the verification of our results and some important observation of the phenomenon in the flow, then we carried out the POD analysis of the DNS flowfield using the method described in the above section, about 30 snapshots were used. Fig. 5 shows the modal energy and cumulative modal energy distributions. Modal energy, which is measured by the eigenvalue, represents the proportion of turbulent energy contained in each corresponding mode. The distribution shows that the first mode is the most dominant, and that only a limited number contain an appreciable fraction of the total energy. By calculating the cumulative modal energy distribution, we can see that 80% of the total energy is contained in approximately first 10 modes. It means that we can use the linear combination of only a few spatial modes to capture most of the turbulence dynamics. This idea has a significant impact both theoretically and in practice. For example, it can be used to study the dynamics in a reduced-dimension space, and design control based on ODEs (ordinary differential equations) rather than PDEs (partial differential equations), obviously, the former is much matured than the latter one.

Fig. 6 is the POD coefficients of the first 4 POD modes. POD coefficients represent how the combination of POD modes varies with time. In Fig. 6, the coefficients exhibit a periodic variation. As we can see, the more energy contained in the mode, the lower its frequency, which means that most of the turbulent energy is linked with structures of large spatial scale and low time frequency. This is observation is consistent with Kolmogrov theory of turbulent energy spectral. Fig. 7 shows the contour of u velocity of the first two POD modes, flooded by normal height. In the
figure, we can observe that before the interaction the main coherent structure is “A-shape” streamwise vortices, and in the interaction region, there are “bump-like” structures, showing that turbulent energy generation is very active in this region. Also, compared with the first mode, the structures in the second mode is finer, this again reflects the energy distribution among different spatial scales.

![Figure 5](image1.png)  
**Fig. 5.** Modal energy and cumulative modal energy

![Figure 6](image2.png)  
**Fig. 6.** POD coefficients of first 4 modes

![Figure 7](image3.png)  
**Fig. 7.** The first POD mode (a) and the second POD mode (b)

5. **Conclusions:**

We performed a POD analysis on a DNS dataset of a SWBLI flowfield. The results show that the incoming boundary layer contains large scale coherent motions, in the form of stream wise elongated regions of low- and high-speed fluid. In the interaction region, the coherent structures are deformed and break up by the shockwave and exhibit a irregular “bump” shape. Distribution of eigenvalues of the corresponding eigenvalue problem shows that most of the turbulent energy can be contained by a few spatial modes. Their combination can reconstruct most of the turbulent dynamics. Analysis of POD modes reflects that distribution of energy among different scales, which is consistent with Kolmogrov theory. The results of POD analysis can be further utilized in turbulence modeling and control design.

REFERENCES

4. Fang J., Li Z. and Lu L. An Optimized Low-Dissipation Monotonicity-Preserving Scheme for Numerical
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